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EMBEDDINGS OF l_p^m INTO l_∞^n, $1 \leqq p \leqq 2$

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ABSTRACT

Isomorphic embeddings of l_1^m into l_∞^n are studied, and for $d(n,k) = \inf\{\|T\|\,\|T^{-1}\|;\ T$ varies over all isomorphic embeddings of $l_1^{[k \log_2 n]}$ into $l_\infty^n\}$ we have that $\lim_{n\to\infty} d(n,k) = \gamma(k)^{-1}$, $k > 1$, where $\gamma(k)$ is the solution of $(1+\gamma)\ln(1+\gamma) + (1-\gamma)\ln(1-\gamma) = k^{-1}\ln 4$.

0. Introduction

The purpose of this paper is to study isomorphic embeddings of l_1^m into l_∞^n. Since l_p^n, $1 < p \leqq 2$, can be 2-isomorphically embedded into $l_1^{[\alpha_p n]}$ for some $\alpha_p > 1$ (cf. [5], [6]) we then automatically get results for embeddings of l_p^m into l_∞^n.

Combining the result of Section 1, in which we consider random embeddings of l_1^m into l_∞^n, with an estimate, given in Theorem 2.1, of $d(n,k) = \inf\{\|T\|\,\|T^{-1}\|;$ T varies over all isomorphic embeddings of $l_1^{[k \log_2 n]}$ into $l_\infty^n\}$[†] from below, we obtain that $\lim_{n\to\infty} d(n,k) = \gamma(k)^{-1}$, $k > 1$ ($d(n,k) \equiv 1$ if $k \leqq 1$). $\gamma(k)$ is the solution of

$$(0.1) \qquad (1+\gamma)\ln(1+\gamma) + (1-\gamma)\ln(1-\gamma) = k^{-1}\ln 4$$

(where $\ln = \log_e$) and satisfies

$$(0.2) \qquad k^{-1/2} < \gamma(k) < (\ln 4)^{1/2} k^{-1/2}, \quad \lim_{k\to\infty} \gamma(k)^2 k = \ln 4.$$

Consequently we get for every $K > 1$ good estimates of the largest dimension of a subspace V of l_∞^n which are K-isomorphic to $l_1^{\dim V}$. As a comparison, estimates this precise do not seem to be known for embeddings of l_2^m into l_p^n, $1 \leqq p \leqq \infty$, studied for example in [2], [3], [5], [7], [8] and a recent paper of V. D. Milman.

All proofs are carried out with the assumption of real scalars. See however the remark following Corollary 2.2.

† Here $[x]$ denotes the integer part of the real number x.

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1. Random matrices

Let $(e_i)_{i=1}^m$ and $(f_j)_{j=1}^n$ be the natural unit vector basis in l_1^m and l_∞^n respectively. Every real matrix $(t_{ij})_{i=1,j=1}^{m,n}$ defines a linear mapping $T_n: l_1^m \to l_\infty^n$ by $T_n e_i = \Sigma_{j=1}^n t_{ij} f_j$, $i = 1, 2, \ldots, m$, and

$$(*) \qquad \| T_n^{-1} \|^{-1} = \inf_{\|b\|_1 = 1} \| T_n b \|_\infty = \inf_{\|b\|_1=1} \max_{1 \leqq j \leqq n} \left| \sum_{i=1}^m b_i t_{ij} \right| .$$

We estimate this from below for a class of random matrices.

THEOREM 1.1. *Let $0 < \beta \leqq 1$, $k > 1$ and $\gamma' < \gamma(k)$ be given. Put $m(n) = [k \log_2 n]$, $n = 2, 3, \ldots$. Assume that $(t_{ij}^n)_{i=1,j=1}^{m(n),n}$, $n = 2, 3, \ldots$, is a sequence of real matrices, the entries of which are independent, symmetric random variables with $\beta \leqq |t_{ij}^n| \leqq 1$. Then there exists $n_0 = n_0(\beta, k, \gamma')$ such that for $n \geqq n_0$*

$$P\left(\inf_{\|b\|_1=1} \max_{1 \leqq j \leqq n} \left| \sum_{i=1}^{m(n)} b_i t_{ij}^n \right| > \beta\gamma' \right) \geqq 1 - \exp(-n^{(1+\log_2 s((\gamma'+\gamma(k))/2)^k)/2})$$

where

$$s(t) = ((1+t)^{(1+t)}(1-t)^{(1-t)})^{-1/2}, \qquad 0 \leqq t \leqq 1. \qquad \square$$

Note that $1 + \log_2 s((\gamma' + \gamma(k))/2)^k > 0$ since

$$2^{-1} = s(\gamma(k))^k < s((\gamma' + \gamma(k))/2)^k.$$

Especially, since

$$\max_{1 \leqq j \leqq n} \left| \sum_{i=1}^{m(n)} b_i t_{ij}^n \right| \leqq \sum_{i=1}^{m(n)} |b_i| = \|b\|_1,$$

we conclude, in view of (*), from the theorem that given $k > 1$ we can, for n large enough, find a subspace $U \subset l_\infty^n$ of dimension $m(n)$ for which $d'(U, l_1^{m(n)})$ is arbitrarily close to $\gamma(k)^{-1}$. Here d' is the Banach–Mazur distance (for isomorphic spaces U and V, $d'(U, V) = \inf\{\|T\| \|T^{-1}\|$; T varies over all isomorphisms of U onto $V\}$).

To prove Theorem 1.1 we need two lemmas, the first of which is a standard lemma on δ-nets of the unit sphere of a finite dimensional Banach space (cf. [5]).

LEMMA 1.2. *Let X be a Banach space of dimension m. Suppose $\delta > 0$ is given. Then the unit sphere $S(X)$ has a δ-net of cardinality $\leqq (1 + 2/\delta)^m$.* $\square$

We will also need

LEMMA 1.3. *Let $(X_i)_{i=1}^m$ be a sequence of real identically distributed random* *variables. Assume that there are constants p and λ such that*

$$P\left(m^{-1}\left|\sum_{i=1}^{m} X_i\right| \geqq \lambda\right) \geqq p.$$

Then

$$P(|X_i| \geqq \lambda) \geqq p/m, \qquad i = 1, 2, \ldots, m. \qquad \square$$

PROOF.

$$p \leqq P\left(m^{-1}\left|\sum_{i=1}^{m} X_i\right| \geqq \lambda\right) \leqq P\left(\max_{1 \leqq i \leqq m} |X_i| \geqq \lambda\right) \leqq \sum_{i=1}^{m} P(|X_i| \geqq \lambda)$$

$$= mP(|X_j| \geqq \lambda) \quad \text{for every } j = 1, 2, \ldots, m. \qquad \blacksquare$$

PROOF OF THEOREM 1.1. Let $(\varepsilon_i)_{i=1}^m$ be a sequence of independent random variables each taking the values $+1$ and -1 with probability $\frac{1}{2}$. Put $\gamma_0 = (2\gamma' + \gamma(k))/3$. Then

$$P\left(m^{-1}\left|\sum_{i=1}^{m} \varepsilon_i\right| > \gamma_0\right) \geqq \binom{m}{[(1+\gamma_0)m/2]} 2^{-m}.$$

By Stirling's formula

$$\binom{m}{[(1+\gamma_0)m/2]} 2^{-m} \sim m^{-1/2}((1+\gamma_0)^{(1+\gamma_0)}(1-\gamma_0)^{(1-\gamma_0)})^{-m/2}.$$

Hence there is $m_1(k, \gamma')$ such that

$$P\left(m^{-1}\left|\sum_{i=1}^{m} \varepsilon_i\right| \geqq \gamma_0\right) \geqq s((\gamma' + \gamma(k))/2)^m, \qquad m \geqq m_1(k, \gamma'). \tag{1.1}$$

Let $\delta = \beta(\gamma(k) - \gamma')/3$. By Lemma 1.2 we can choose a δ-net N of cardinality $\leqq (3/\delta)^{m(n)}$ on the unit sphere of $l_1^{m(n)}$. We have

$$\begin{aligned} P\left(\inf_{b \in N} \max_{1 \leqq j \leqq n}\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| \geqq \beta\gamma_0\right) &= 1 - P\left(\inf_{b \in N} \max_{1 \leqq j \leqq n}\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| < \beta\gamma_0\right) \\ &\geqq 1 - \sum_{b \in N} P\left(\max_{1 \leqq j \leqq n}\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| < \beta\gamma_0\right) \qquad (1.2) \\ &= 1 - \sum_{b \in N}^{n} \prod_{j=1}^{n}\left(1 - P\left(\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| \geqq \beta\gamma_0\right)\right). \end{aligned}$$

For simplicity, let us write t_i in place of t_{ij}^n for any j and n fixed. Then, since $\beta \leqq \sum_{k=1}^{m(n)} |b_k t_k|$, we get

$$P\left(\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| \geqq \beta\gamma_0\right) = P\left(\left|\sum_{i=1}^{m(n)} b_i t_i\right| \geqq \beta\gamma_0\right)$$

$$\geqq P\left(\left|\sum_{i=1}^{m(n)} \frac{b_i \mid t_i \mid}{\left(\sum_{k=1}^{m(n)} \mid b_k t_k \mid\right)} \operatorname{sign}(t_i)\right| \geqq \gamma_0\right).$$

Now make the assumption that all $b_i \geqq 0$ and put $d_i = b_i \mid t_i \mid / (\sum_{k=1}^{m(n)} \mid b_k t_k \mid)$. Define π_i, $1 \leqq i \leqq m(n)$, by $\pi_i(j) = (i + j - 1)\operatorname{mod}(m(n))$, $1 \leqq j \leqq m(n)$. Then define a sequence of (dependent) identically distributed random variables by

$$X_i = \sum_{j=1}^{m(n)} d_{\pi_i(j)} \operatorname{sign}(t_j), \qquad 1 \leqq i \leqq m(n).$$

Then, since $b_i \geqq 0$ we get

$$\sum_{i=1}^{m(n)} X_i = \sum_{i=1}^{m(n)} \operatorname{sign}(t_i).$$

By (1.1) we have

$$\begin{aligned} P\left(m(n)^{-1}\left|\sum_{i=1}^{m(n)} X_i\right| \geqq \gamma_0\right) &= P\left(m(n)^{-1}\left|\sum_{i=1}^{m(n)} \operatorname{sign}(t_i)\right| \geqq \gamma_0\right) \\ &\geqq s((\gamma' + \gamma(k))/2)^{m(n)}, \quad m(n) \geqq m_1(k, \gamma'). \end{aligned} \tag{1.3}$$

Hence an application of Lemma 1.3 yields

$$\begin{aligned} P\left(\left|\sum_{i=1}^{m(n)} \frac{b_i \mid t_i \mid}{\left(\sum_{k=1}^{m(n)} \mid b_k t_k \mid\right)} \operatorname{sign}(t_i)\right| \geqq \gamma_0\right) &= P(\mid X_1 \mid \geqq \gamma_0) \\ &\geqq m(n)^{-1} s((\gamma' + \gamma(k))/2)^{m(n)}. \end{aligned} \tag{1.4}$$

Thus, returning to (1.2), we get that

$$P\left(\left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| \geqq \beta\gamma_0\right) \geqq m(n)^{-1} s((\gamma' + \gamma(k))/2)^{m(n)}, \quad m(n) \geqq m_1(k, \gamma')$$

holds if $b_i \geqq 0$, $1 \leqq i \leqq m(n)$. By symmetry, this holds for all $b \in N$. Substituting (1.4) into (1.2) we hence get that

$$\begin{aligned} P\left(\inf_{b \in N} \max_{1 \leqq j \leqq n} \left|\sum_{i=1}^{m(n)} b_i t_{ij}^n\right| \geqq \beta\gamma_0\right) &\geqq 1 - (3/\delta)^{m(n)}(1 - m(n)^{-1} s((\gamma' + \gamma(k))/2)^{m(n)})^n \\ &\geqq 1 - (3/\delta)^{m(n)} \exp(-s((\gamma' + \gamma(k))/2)^{m(n)} n / m(n)) \\ &\geqq 1 - (3/\delta)^{m(n)} \exp(-m(n)^{-1} n^{(1 + \log_2 s((\gamma' + \gamma(k))/2)^k)}) \\ &\geqq 1 - \exp(-n^{(1 + \log_2 s((\gamma' + \gamma(k))/2)^k)/2}) \end{aligned}$$

if n is large enough depending on β, k and γ'. Since this holds for the δ-net N where $\delta = \beta(\gamma(k) - \gamma')/3$, the proof is completed. ■

COROLLARY 1.4. *Let $k > 1$ be given and put $m(n) = [k \log_2 n]$, $n = 2, 3, \dots$. Assume that $T_n : l_1^{m(n)} \to l_\infty^n$, $n = 2, 3, \dots$, are represented by random matrices $(t_{ij}^n)_{i=1, j=1}^{m(n), n}$, each of which satisfies the hypotheses of Theorem 1.1 with $\beta = 1$. Then*

$$\limsup_{n \to \infty} \| T_n \| \| T_n^{-1} \| \leq \gamma(k)^{-1} \quad \text{a.s.} \qquad \square$$

PROOF. Let $k > 1$. Choose an increasing sequence $(\lambda_l)_{l=1}^\infty$ with $\lim_{l \to \infty} \lambda_l = \gamma(k)$. Then

$$\delta_l = 1 + \log_2 s((\lambda_l + \gamma(k))/2)^k > 0.$$

Theorem 1.1 gives

$$\sum_{n=2}^\infty P(\| T_n \| \| T_n^{-1} \| > \lambda_l^{-1}) = \sum_{n=2}^\infty P\left(\inf_{\|b\|_1 = 1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{m(n)} b_i t_{ij}^n \right| < \lambda_l \right)$$

$$\leq \sum_{n=2}^{n_0} P\left(\inf_{\|b\|_1 = 1} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{m(n)} b_i t_{ij}^n \right| < \lambda_l \right)$$

$$+ \sum_{n=n_0+1}^\infty e^{-n^{\delta_l}} < \infty \qquad \text{for each } l = 1, 2, \dots.$$

By the Borel–Cantelli lemma, the probability that $\| T_n \| \| T_n^{-1} \| > \lambda_l^{-1}$ for infinitely many n is zero. Hence

$$P\left(\limsup_{n \to \infty} \| T_n \| \| T_n^{-1} \| > \lambda_l^{-1} \right) = 0 \qquad \text{for each } l = 1, 2, \dots.$$

We conclude that

$$P\left(\limsup_{n \to \infty} \| T_n \| \| T_n^{-1} \| \leq \gamma(k)^{-1} \right) = 1. \qquad ■$$

2. Estimates of $d(n, k)$ from below

Most embeddings of $l_1^{m(n)}$ into l_∞^n by means of matrices with entries ± 1, as in Section 1, are close to being best possible. We have

THEOREM 2.1. *Let $k > 1$ and $\varepsilon < (1 - \gamma(k))/2$ be given. Set $m(n) = [k \log_2 n]$, $n = 2, 3, \dots$. Then for n so large that*

$$2m(n)n \sum_{i=0}^{[\varepsilon m(n)]} \binom{m(n)}{i} < 2^{m(n)}/2 \tag{2.1}$$

holds, we have

$$d(n,k) \geqq (1-2\varepsilon+2/m(n))^{-1}. \qquad \square$$

COROLLARY 2.2. *Let $k>1$. Then*

$$\lim_{n\to\infty} d(n,k)=\gamma(k)^{-1}. \qquad \square$$

PROOF. This follows immediately from Theorem 2.1 and Corollary 1.4. ■

REMARK. In the case of complex scalars one obtains, by separating real and imaginary parts, that $(2\gamma(k))^{-1} \leqq \underline{\lim}_{n\to\infty} d(n,k) \leqq \overline{\lim}_{n\to\infty} d(n,k) \leqq 2\gamma(k)^{-1}$.

PROOF OF THEOREM 2.1. Let $k>1$ and $\varepsilon<(1-\gamma(k))/2$. For $T: l_1^{m(n)} \to l_\infty^n$ represented by a matrix (a_{ij}) such that $Te_i=\sum_{j=1}^n a_{ij}f_j$, $i=1,2,\dots,m(n)$, we have

$$\|T\|=\sup_{\|b\|_1=1}\left\|T\left(\sum_{i=1}^{m(n)} b_ie_i\right)\right\|_\infty=\max_{i,j}|a_{i,j}|.$$

Thus

$$d(n,k)=\inf\{\|T\|;\ T \text{ varies over all embeddings of } l_1^{m(n)} \text{ into } l_\infty^n \text{ with } \|T^{-1}\|=1\}$$

$$(2.2)\qquad =\inf\left\{\max_{i,j}|a_{ij}|;\ \inf_{\|b\|_1=1}\max_{1\leqq j\leqq n}\left|\sum_{i=1}^{m(n)} b_ia_{ij}\right|=1\right\}$$

$$\geqq \inf\left\{\max_{i,j}|a_{ij}|;\ \inf_{\delta_i=\pm1}\max_{1\leqq j\leqq n}\left|\sum_{i=1}^{m(n)} \delta_ia_{ij}\right|=m(n)\right\}.$$

To estimate this, let (a_{ij}) be any matrix that satisfies

$$(2.3)\qquad \inf_{\delta_i=\pm1}\max_{1\leqq j\leqq n}\left|\sum_{i=1}^{m(n)} \delta_ia_{ij}\right|=m(n).$$

We now want to show the existence of signs $(\delta_i)_{i=1}^{m(n)}$, such that the terms in the sum in (2.3), to a large extent, cancel, thereby forcing $\max_{i,j}|a_{ij}|$ to be large. To accomplish this, we decompose (a_{ij}) into a sum of matrices as follows. Let d_j^l, $\varphi_{ij}^l \subset \mathbf{R}$, $l=1,2,\dots,2m(n)$, be such that

(i) $d_j^l \geqq 0$ and $\varphi_{ij}^l=\pm1$,
(ii) $a_{ij}=\sum_{l=1}^{2m(n)} d_j^l\varphi_{ij}^l$ and
(iii) $\max_{1\leqq i\leqq m(n)}|a_{ij}|=\sum_{l=1}^{2m(n)} d_j^l$.

It is not difficult to see that there exists such a choice of numbers.

$\{-1,1\}^{m(n)}$ becomes a metric space M when equipped with the metric

$$\operatorname{dist}(\psi,\theta)=2^{-1}\sum_{i=1}^{m(n)}|\psi_i-\theta_i|, \qquad \psi,\theta\in\{-1,1\}^{m(n)}.$$

Note that each $\psi \in M$ defines an isometry $I_\psi : M \to M$ by

$$(I_\psi(\theta))_i = \psi_i \theta_i, \qquad 1 \leqq i \leqq m(n), \quad \theta \in M.$$

For each $\psi \in M$, let $V_\varepsilon(\psi) = \{\theta \in M;\ \mathrm{dist}(\psi, \theta) \leqq \varepsilon m(n)\}$ be the $\varepsilon m(n)$-neighbourhood of ψ.

The sequences $\varphi_j^l = (\varphi_{ij}^l)_{i=1}^{m(n)}$, $1 \leqq j \leqq n$, $1 \leqq l \leqq 2m(n)$ belong to M and $\bigcup_{j,l} V_\varepsilon(\varphi_j^l)$ covers at most

$$2m(n)n \sum_{i=0}^{[\varepsilon m(n)]} \binom{m(n)}{i} \text{ points in } M.$$

Suppose n is so large that (2.1) holds. Then there exists $\psi^0 \in M$ such that $-\psi^0, \psi^0 \notin \bigcup_{j,l} V_\varepsilon(\varphi_j^l)$. Since I_{ψ^0} is an isometry on M we get that

$$(1)_{i=1}^{m(n)}, (-1)_{i=1}^{m(n)} \notin \bigcup_{j,l} V_\varepsilon(I_{\psi^0}(\varphi_j^l)). \tag{2.4}$$

Estimating the left side in (2.3) we now get that

$$\begin{aligned}
m(n) &= \inf_{\delta_i = \pm 1} \max_{1 \leqq j \leqq n} \left| \sum_{i=1}^{m(n)} \delta_i a_{ij} \right| \\
&\leqq \max_{1 \leqq j \leqq n} \left| \sum_{l=1}^{2m(n)} d_j^l \sum_{i=1}^{m(n)} \psi_i^0 \varphi_{ij}^l \right| \\
&\leqq \max_{1 \leqq j \leqq n} \sum_{l=1}^{2m(n)} d_j^l \left| \sum_{i=1}^{m(n)} \psi_i^0 \varphi_{ij}^l \right| && \text{(by (2.4))} \\
&\leqq \max_{1 \leqq j \leqq n} \sum_{l=1}^{2m(n)} d_j^l (m(n) - 2[\varepsilon m(n)]) \\
&= \max_{1 \leqq j \leqq n} \max_{1 \leqq i \leqq m(n)} |a_{ij}| (m(n) - 2[\varepsilon m(n)])
\end{aligned}$$

which implies that

$$\max_{i,j} |a_{ij}| \geqq m(n)(m(n) - 2[\varepsilon m(n)])^{-1} \geqq (1 - 2\varepsilon + 2/m(n))^{-1}.$$

Hence, by (2.2), we get that

$$d(n,k) \geqq (1 - 2\varepsilon + 2/m(n))^{-1}. \qquad \blacksquare$$

In order to consider isomorphic embeddings of l_p^m into l_∞^n, $1 \leqq p \leqq 2$, we will need

PROPOSITION 2.3. *There exists a constant $C > 0$ such that if $E \subset l_\infty^n$ is a* *subspace with* $\dim E = \nu$, *then*

$$d'(E, l_2^\nu) \geqq C(\nu/\ln n)^{1/2}. \qquad \square$$

For a proof we refer to [4]. A consequence of Proposition 2.3 is that there exists a constant $C_0 > 0$ such that for all n

$$d(n,k) \geqq C_0 k^{1/2}. \tag{2.5}$$

To prove (2.5) we proceed as follows. Let $1 \leqq p < 2$. Since $l_2^{\alpha_p m}$ embeds 2-isomorphically into l_p^m for some $\alpha_p > 0$ (cf. [5]), we can for any positive integer n find $S_n : l_2^{\alpha_p m(n)} \to l_p^{m(n)}$ with $\| S_n \| \, \| S_n^{-1} \| \leqq 2$. Suppose $T_n : l_p^{m(n)} \to l_\infty^n$ is injective. Proposition 2.3 applied with

$$E = T_n \circ S_n (l_2^{\alpha_p m(n)})$$

gives that

$$2 \| T_n \| \, \| T_n^{-1} \| \geqq \| T_n \circ S_n \| \, \| (T_n \circ S_n)^{-1} \| \geqq C(\alpha_p [k \log_2 n]/\ln n)^{1/2}$$

which implies that $\| T_n \| \, \| T_n^{-1} \| \geqq C_p k^{1/2}$, for some constant $C_p > 0$ depending only on p. Thus (2.5) holds. The preceding argument also proves the latter part of the next result.

Proposition 2.4. *Let* $1 \leqq p \leqq 2$. *Then there exist positive constants* K_p *and* C_p *such that if* $k > 1$ *and* $m(n) = [k \log_2 n]$, $n = 2, 3, \ldots$, *then*

(i) *for* n *large enough depending on* k, *there exist isomorphic embeddings* $T_n : l_p^{m(n)} \to l_\infty^n$ *with* $\| T_n \| \, \| T_n^{-1} \| \leqq K_p k^{1/2}$,

(ii) *for any linear and injective mapping* $T_n : l_p^{m(n)} \to l_\infty^n$ *we have* $\| T_n \| \, \| T_n^{-1} \| \geqq C_p k^{1/2}$. $\square$

Proof. It remains to prove (i) for $1 < p \leqq 2$. Since $l_p^{m(n)}$ embeds 2-isomorphically into $l_1^{\alpha_p m(n)}$ for some $\alpha_p > 1$ (cf. [5], [6]), we can find $S_n : l_p^{m(n)} \to l_1^{\alpha_p m(n)}$ with $\| S_n \| \, \| S_n^{-1} \| \leqq 2$. Applying Theorem 1.1 with $\beta = 1$ and $\gamma' = k^{-1/2}$ we get, for n large enough depending on k, a mapping $T_n : l_1^{\alpha_p m(n)} \to l_\infty^n$ with $\| T_n \| \, \| T_n^{-1} \| < (\alpha k)^{1/2}$. Hence $T_n \circ S_n : l_p^{m(n)} \to l_\infty^n$ satisfies

$$\| T_n \circ S_n \| \, \| (T_n \circ S_n)^{-1} \| \leqq 2\alpha_p^{1/2} k^{1/2} \leqq K_p k^{1/2}. \qquad \blacksquare$$

The results in Sections 1 and 2 show that, for n large enough, most matrices with entries ± 1 represent good embeddings of $l_1^{m(n)}$ into l_∞^n. A related result holds for embeddings of $l_2^{m(n)}$ into l_∞^n.

Proposition 2.5. *Let* $0 < p < 1$ *and* $k > 1$ *be given. Then there exist* $c > 1$ *and* $n_0 \in \mathbf{N}$, *where* c *only depends on* p, *such that the following holds. Let* $m(n) =$

$[k \log_2 n]$, $n = 2,3,\ldots$, and assume that $(s_{ij})_{i=1,j=1}^{m(n)[cm(n)]}$ and $(t_{ij})_{i=1,j=1}^{[cm(n)],n}$, $n \geqq n_0$, are random matrices with independent entries each taking the values $+1$ and -1 with probability $\frac{1}{2}$. Then, if $S_n : l_2^{m(n)} \to l_1^{[cm(n)]}$ and $T_n : l_1^{[cm(n)]} \to l_\infty^n$ denote the corresponding linear mappings, we have for $T_n \circ S_n : l_2^{m(n)} \to l_\infty^n$ that

$$P(\| T_n \circ S_n \| \, \| (T_n \circ S_n)^{-1} \| \leqq 16 c^{1/2} k^{1/2}) > p. \qquad \square$$

PROOF. From the proof of Theorem 1 in [8] it follows that, given $0 < p < 1$, there exists $c > 1$ such that for a proportion larger than $(1+p)/2$ of all $m(n) \times [cm(n)]$ matrices with entries ± 1, we have for the corresponding linear mapping $S_n : l_2^{m(n)} \to l_1^{[cm(n)]}$ that $\| S_n \| \, \| S_n^{-1} \| \leqq 16$. An application of Theorem 1.1 now finishes the proof. ■

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